Kinetic Models for Magnetized Plasmas

Alexandre Rege
Supervisors: Frédérique Charles and Bruno Després

Laboratoire Jacques-Louis Lions

Thesis Defense
October 18, 2021

Outline

Introducing the magnetized Vlasov–Poisson system

The Bernstein–Landau paradox

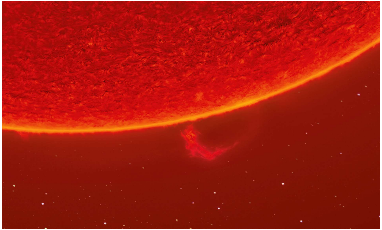
Propagation of velocity moments and uniqueness for the magnetized Vlasov–Poisson system

Plasmas

Definition (Plasma)

Plasma is a state of matter, characterized by an important ionization of particles.

- Term "plasma" first used by Irving Langmuir [1] to describe an ionized gas.
- Plasma represents more than 99% of ordinary matter in the universe.



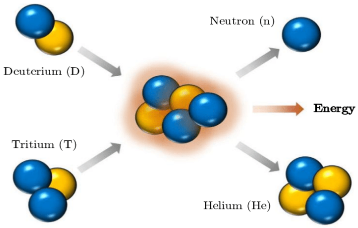
[1] I. Langmuir, Oscillations in Ionized Gases, Proceedings of the National Academy of Science, 14:627–637, 1928.

Nuclear fusion

Definition (Nuclear fusion)

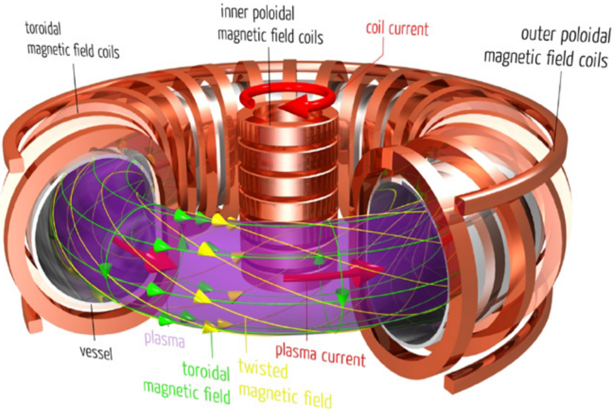
Combination of two atomic nuclei to form one atomic nuclei.

- Fusion of light nuclei releases enormous amounts of energy.
- This reaction takes place in plasmas under certain conditions: very high temperature and high enough particle density.
- One approach to harness nuclear fusion of light nuclei is via magnetic confinement.



Magnetic confinement fusion

- Use an intense external (not self-induced) magnetic field B to confine the hot plasma.
- Feasibility of controlled nuclear fusion: ITER tokamak under construction in Cadarache (south of France).



Kinetic formalism and the Vlasov equation

Trajectory $(X(t), V(t))$ of one particle of mass m subject to a force $F(t)$ given by:

$$\begin{cases} \dot{X}(t) = V(t), \\ \dot{V}(t) = \frac{1}{m} F(t). \end{cases} \quad (1)$$

Large number of identical particles allows for a kinetic description of the system. $f(t, x, v)$ is the number density of particles which are located at the position x and have velocity v at time t and satisfies:

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) + \frac{1}{m} F(t) \cdot \nabla_v f(t, x, v) = 0 \quad (2)$$

with $(t, x, v) \in \mathbb{R}^+ \times X^3 \times \mathbb{R}^3$, $X = \mathbb{R}$ or $X = \mathbb{T}$.

The magnetized Vlasov–Poisson system for electrons

- At the time scale of electrons: ions are static $\Longrightarrow f_{ion}$ is constant.
- At the time scale of ions: electrons are at thermal equilibrium $\Longrightarrow f_{electron} =$ Maxwell–Boltzmann type distribution.

At the time scale of electrons, we have:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q_e}{m_e}(E + v \wedge B) \cdot \nabla_v f = 0, \\ \mathrm{div}_x E(t,x) = \frac{q_{ion}}{\epsilon_0} \int_{\mathbb{R}^3} f_{ion}(x,v) dx dv + \frac{q_e}{\epsilon_0} \int_{\mathbb{R}^3} f(t,x,v) dx dv, \end{cases} \tag{3}$$

with $f \equiv f(t,x,v)$ the distribution function of electrons, $f_{ion}(x,v)$ the constant ion distribution, $E \equiv E(t,x)$ the self-consistent electric field and $B \equiv B(t,x)$ the **given** external magnetic field.

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Landau damping

Exponential decrease in time of longitudinal waves in plasma.

- First predicted by Landau with the linearized Vlasov-Poisson system where he showed damping of the electric field E.[2]

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q}{m} E \cdot \nabla_v f_0 = 0, \\ \partial_x E = \frac{q}{\epsilon_0} \int f dv_1 dv_2, \end{cases} \tag{4}$$

  with $f = f(t,x,v)$, $(t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$ the distribution of electrons and $f_0 = e^{-\frac{|v|^2}{2}}$ the equilibrium Maxwellian distribution.

- Using the Fourier–Laplace transform, Landau showed the damping of the electric field

$$|E_k(t)| \leq Ce^{-\alpha_k t}. \tag{5}$$

- Irreversible behavior observed in a reversible in time system.
- Recently extended to the nonlinear setting by Mouhot and Villani [3].

[2] L. Landau, On the vibration of the electronic plasma, J. Phys. USSR, 1946.
[3] C. Mouhot and C. Villani, On Landau damping, Acta Math., 2011.

The paradox

The Bernstein-Landau paradox

"In unmagnetized plasmas, waves exhibit Landau Damping, while in magnetized plasmas, waves perpendicular to the magnetic field are exactly undamped, independently of the strength of the magnetic field".

- Magnetized plasmas first studied by Bernstein [4].
- Other physical works [5] have highlighted a discontinuity between the theory of unmagnetized plasmas and the theory of magnetized plasmas, thus speaking of a paradox.
- Very few mathematical papers, recently studied in [6] using Fourier–Laplace in a 3d-3v setting.

[4] I. Bernstein, Waves in a Plasma in a Magnetic Field, Phy. Review, 1958.

[5] A. I. Sukhorukov and P. Stubbe, On the Bernstein-Landau paradox, Phy. of Plasmas, 1997.

[6] J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, J. Stat. Phys., 2020.

Numerical illustration of the influence of B: magnetic recurrence

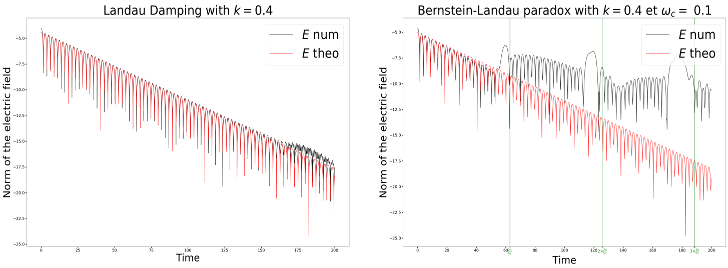


Figure: Damped and undamped electric field

This is a magnetic recurrence [7].

[7] Not a numerical recurrence *Recurrence phenomenon for Vlasov-Poisson simulations on regular finite element mesh*, M. Mehrenberger, L. Navoret, N. Pham, Commun. Comput. Phys., 2020.

The 1d-2v framework

The 1d-2v magnetized Vlasov-Poisson system

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \underbrace{\omega \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2} \right) f}_{= -v \wedge B} = 0, \\ \partial_x E = 2\pi - \int f dv_1 dv_2. \end{cases} \tag{6}$$

- The unknowns are the density of electrons $f(t, x, v_1, v_2) \geq 0$ and the electric field $E(t, x)$.
- The domain is $\Omega = \mathbb{T} \times \mathbb{R}^2, \quad \mathbb{T} = [0\,, 2\pi]_{\text{per}}$ is the 1D-torus.
- Here $\omega > 0$ is the constant cyclotron frequency for electrons $(B = (0, 0, \omega))$.
- $\frac{q_e}{m_e}$ normalized to -1.
- Ions are considered as a motionless background of neutralizing positive charge.

Linearized system

We linearize (6) by writing

$$f = f_0 + \varepsilon\sqrt{f_0}u + O(\varepsilon^2) \text{ and } E = \varepsilon F + O(\varepsilon^2).$$

where $(f_0, E_0) = (\exp(-\frac{v_1^2+v_2^2}{2}), 0)$ is a stationary solution of (6).

Linearized Vlasov–Poisson with magnetic field

$$\begin{cases} \partial_t u + v_1\partial_x u + Fv_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega\left(-v_2\partial_{v_1} + v_1\partial_{v_2}\right)u = 0, \\ \partial_x F = -\int ue^{-\frac{v_1^2+v_2^2}{4}} dv_1 dv_2. \end{cases} \tag{7}$$

- $\int ue^{-\frac{v_1^2+v_2^2}{4}} dx dv_1 dv_2 = 0$. (total mass equal zero)
- $\int F dx = 0$. (F is derived from a potential)

Reformulation with the Ampère equation

We can rewrite the system with the Ampère equation because both systems are equivalent.

The linear 1d-2v Vlasov–Ampère system

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega(-v_2 \partial_{v_1} + v_1 \partial_{v_2}) u = 0, \\ \partial_{\mathbf{t}} \mathbf{F} = \mathbf{1}^* \int \mathbf{u} e^{-\frac{\mathbf{v_1^2+v_2^2}}{4}} \mathbf{v_1 dv_1 dv_2}. \end{cases} \tag{8}$$

$$\text{with } 1^* g(x) = g(x) - \frac{1}{2\pi} \int_{\mathbb{T}} g(x) dx.$$

A self-adjoint Vlasov–Ampère operator

Final formulation

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = iH \begin{pmatrix} u \\ F \end{pmatrix}, H = i \left(\begin{array}{c|c} \underbrace{v_1 \partial_x + \omega(v_2 \partial_{v_1} - v_1 \partial_{v_2})}_{:=H_0} & v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ \hline -1^* \int v_1 e^{-\frac{v_1^2+v_2^2}{4}} \cdot dv_1 dv_2 & 0 \end{array} \right).$$

$$\mathcal{H} = \underbrace{(L^2(\mathbb{T} \times \mathbb{R}^2) \cap \left\{ \int u \sqrt{f_0} dx dv_1 dv_2 = 0 \right\})}_{=L_0^2(\mathbb{T}\times\mathbb{R}^2)} \times \underbrace{(L^2(\mathbb{T}) \cap \left\{ \int F dx = 0 \right\})}_{=L_0^2(\mathbb{T})}$$

and H a **self-adjoint** operator with domain $D(H) := D(H_0) \times L_0^2(\mathbb{T})$.

Spectral properties of Vlasov equations

- For a general self-adjoint operator H with an ambient Hilbert space $\mathcal{H}$: then we have the following decomposition of $\mathcal{H}$ in terms of the spectrum of H: $\mathcal{H} = \mathcal{H}_H^{ac} \oplus \mathcal{H}_H^{sc} \oplus \mathcal{H}_H^{pp}$. [8] [9]
- Spectrum of the Vlasov–Ampère operator with $\omega = 0$ studied in [10] [11]: the operator has only absolutely continuous spectrum and a kernel (as expected) $\mathcal{H} = \ker H \oplus \mathcal{H}_H^{ac}$.

[8] T. Kato, Perturbation theory for linear operators, 1966.

[9] D.R. Yafaev, Scattering theory: Some old and new problems, 2000.

[10] B. Després, Symmetrization of Vlasov–Poisson Equations, SIAM J. Math. Anal., 2014.

[11] B. Després, Trace class properties of the linear Vlasov-Poisson equation, J. of Spectral Theory, 2021.

Discrete spectrum for the Vlasov–Ampère operator

Theorem (Charles, Després, R., Weder [12])

We have completeness of the eigenspaces:

$$\mathcal{H} = \mathcal{H}_H^{pp},$$

and the eigenvalues of H are 0, $m\omega$ and λ_m, $m \neq 0$.

Two different proofs:

- Direct computations.
- Weyl theorem on the invariance of the essential spectrum.

Explanation for the Bernstein–Landau paradox

The magnetic term $(v \wedge B) \cdot \nabla_v f$ can be seen as a perturbation that modifies the domain of the Vlasov–Ampère operator.

[12] F. Charles, B. Després, A. Rege and R. Weder, The magnetized Vlasov–Ampère system and the Bernstein–Landau paradox, J. Stat. Phys., 2021.

Numerical illustration: initialization

Objective: compare the numerical and theoretical solutions of Vlasov-Ampère when initializing with an eigenvector.

- We consider an eigenvector $(w_{n,m}\,, F_n)$ associated to the Fourier mode $n \neq 0$ and the eigenvalue λ_m, so the solution $(u\,, F)$ is given by

$$(u\,, F)(t) = e^{i\lambda_m t}\,(w_{n,m}\,, F_n)\,.$$

- $w_{n,m}$ and F_n are given by

$$w_{n,m} = e^{in(x-\frac{v_2}{\omega})} e^{-\frac{r^2}{4}} \sum_{p\in\mathbb{Z}^*} \frac{p\omega}{p\omega+\lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega}\right) \text{ and } F_n = -ine^{inx}.$$

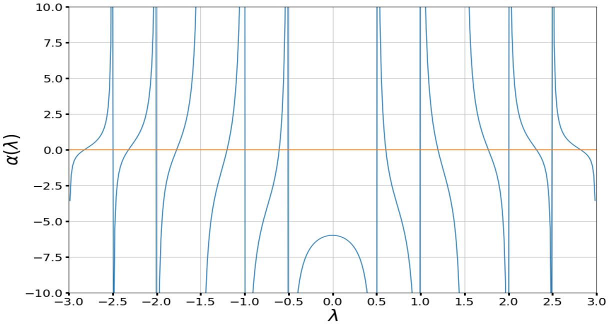
- λ_m is one of the roots of a secular equation given by:

$$\alpha(\lambda) = -1 - \frac{2\pi}{n^2} \sum_{m\in\mathbb{Z}^*} \frac{m\omega}{m\omega+\lambda} \int_0^\infty e^{-\frac{r^2}{2}} J_m\left(\frac{nr}{\omega}\right)^2 rdr = 0.$$

Numerical illustration: secular equation

α has a unique root in

- $]m\omega\,,(m+1)\omega[$ for $m \geq 1$,
- $](m-1)\omega\,,m\omega[$ for $m \leq -1$.



For $(n, m) = (1, 2)$, we compute $\lambda_2 \approx 1.19928$.

Numerical illustration: results

Simulations with a classical "backward" semi-lagrangian scheme, $N_x = 33$, $N_{v_1} = N_{v_2} = 63$, $L_x = 2\pi$, $L_{v_1} = L_{v_2} = 10$, $\omega = 0.5$, $n = 1$, and $\mathbf{T_f} = \frac{\pi}{2\lambda_m}$.

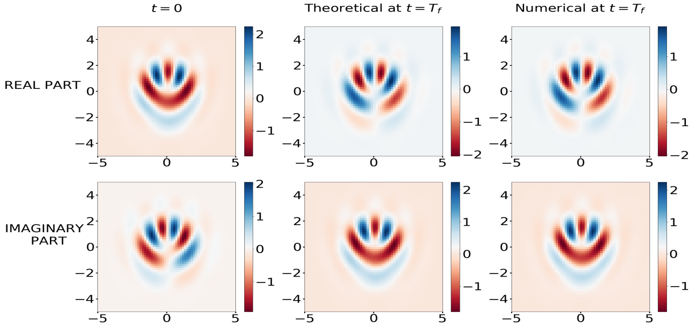


Figure: Real and imaginary parts of u in V1-V2 plane for $x = 0$.

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Back to the nonlinear system in $\mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \wedge B) \cdot \nabla_v f = 0, \\ E(t,x) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|} f(t,y,v) dv dy. \end{cases} \qquad \text{(VPwB)}$$

Energy of the system and macroscopic density ρ

$$\mathcal{E}(t) := \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(t,x,v) dx dv + \frac{1}{2} \int_{\mathbb{R}^3} |E(t,x)|^2 dx,$$

$$\rho(t,x) := \int_{\mathbb{R}^3} f(t,x,v) dv.$$

Results on existence of solutions in the unmagnetized case:

- Existence of weak solutions [Arsenev, 1975]
- Small initial data [Bardos, Degond, 1985]
- Existence of smooth solutions [Pfaffelmoser, 1992]
- Propagation of velocity moments [Lions, Perthame, 1991]

We will first consider a constant B

$$B = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}.$$

Differential inequality on M_k

Let $k \geq 0$, the velocity moment of order k is defined by

$$M_k(t) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t,x,v) dvdx.$$

Propagation of velocity moments:

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x,v) dvdx < \infty \Longrightarrow \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t,x,v) dvdx < \infty$$

$$\begin{aligned}
|\frac{d}{dt} M_k(t)| &= |\iint |v|^k (-v \cdot \nabla_x f - (E + v \wedge B) \cdot \nabla_v f) dvdx|, \\
&= |\iint |v|^k \, \mathrm{div_v} \left((E + v \wedge B) f \right) dvdx|, \\
&= |\iint k|v|^{k-2} v \cdot (E + v \wedge B) f dvdx|, \\
&\leq C \left\| E(t) \right\|_{k+3} M_k(t)^{\frac{k+2}{k+3}}.
\end{aligned}$$

Next step: we need to control of $\|E(t)\|_{k+3}$ with $M_k(t)^{\alpha}$ with $\alpha \leq \frac{1}{k+3}$.

A representation formula for ρ

$$\partial_t f + v \cdot \nabla_x f + (v \wedge B) \cdot \nabla_v f = -E \cdot \nabla_v f$$

$$\begin{cases} \dfrac{d}{ds}\left(X(s), V(s)\right) = \left(V(s), V(s) \wedge B\right) = \left(V(s), (\omega V_2(s), -\omega V_1(s), 0)\right), \\ (X(t), V(t)) = (x, v), \end{cases}$$

$$\begin{cases} V(s) = \begin{pmatrix} v_1 \cos(\omega(s-t)) + v_2 \sin(\omega(s-t)) \\ -v_1 \sin(\omega(s-t)) + v_2 \cos(\omega(s-t)) \\ v_3 \end{pmatrix}, \\ X(s) = \begin{pmatrix} x_1 + \frac{v_1}{\omega} \sin(\omega(s-t)) + \frac{v_2}{\omega}(1 - \cos(\omega(s-t))) \\ x_2 + \frac{v_1}{\omega}(\cos(\omega(s-t)) - 1) + \frac{v_2}{\omega} \sin(\omega(s-t)) \\ x_3 + v_3(s-t)) \end{pmatrix}, \end{cases} \tag{9}$$

$$\rho(t,x) = \underbrace{\int_v f^{in}(X(0), V(0)) dv}_{=:\rho_0(t,x)} + \mathrm{div}_x \int_0^t \underbrace{\int_v (f H_t)(s, X(s), V(s))\, dv}_{=:\sigma(s,t,x)} ds.$$

Singularities at multiples of the cyclotron period

$$E(t,x) = -\left(\nabla \frac{1}{|\cdot|} \star \rho\right)(t,x) = E^0(t,x) + \tilde{E}(t,x),$$

$$\|E(t)\|_{k+3} \leq \left\|E^0(t)\right\|_{k+3} + \int_0^t \|\sigma(s,t,x)\|_{k+3}\, ds,$$

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C\frac{\sqrt{2}}{s}\left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} M_k(t-s)^{\frac{1}{k+3}}. \qquad (10)$$

Proposition (Propagation of moments on a finite interval)

For all $0 \leq t \leq T_\omega := \frac{\pi}{\omega}$ *we have*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)dxdv \leq C < +\infty,$$

with $C = C(k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

Propagation of moments for all time

We have that

- $\left\|f^{in}\right\|_1 = \|f(T_\omega)\|_1$ and $\left\|f^{in}\right\|_\infty = \|f(T_\omega)\|_\infty$,
- $\mathcal{E}(T_\omega) \leq \mathcal{E}_{in}$,
- $M_k(f(T_\omega)) \leq C(k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

This means $f(T_\omega)$ verifies the same assumptions as $f^{in} \implies$ we can show propagation of moments for all time by induction.

Additional results

- Propagation of the regularity of f^{in}.
- Uniqueness if $\rho \in L^\infty([0\,,\,T] \times \mathbb{R}^3)$ [13].

[13] G. Loeper, Uniqueness of the solution to the Vlasov–Poisson system with bounded density, 2006.

Propagation of velocity moments for (VPwB)

Theorem (R. [14])

Let $k_0 > 3, T > 0, f^{in} = f^{in}(x,v) \geq 0$ *a.e. with* $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{k_0} f^{in} dxdv < \infty. \tag{11}$$

Then there exists $C > 0$ *and a weak solution* f *to* (VPwB) *with* $B = (0,0,\omega)$ *such that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{k_0} f(t,x,v) dxdv \leq C < +\infty, \quad 0 \leq t \leq T \tag{12}$$

with C *that depends on*

$$T, k_0, \omega, \mathcal{E}_{in}, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}). \tag{13}$$

[14] A. Rege, The Vlasov–Poisson system with a uniform magnetic field: propagation of moments and regularity, SIAM J. Math. Anal., 2021.

Uniqueness in the unmagnetized case

Theorem (Miot [15])

Let $T > 0$. Then there exists at most one solution $f \in L^\infty([0\,,T], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ to Vlasov–Poisson such that

$$\sup_{[0\,,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{14}$$

New uniqueness criterion which allows for solutions with unbounded ρ.

[15] E. Miot., A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system, Comm. Math. Phys., 2016.

Characteristics

Now we have $B := B(t,x)$ such that

$$B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3). \tag{15}$$

$$\begin{cases} \dfrac{d}{ds} X(s;t,x,v) = V(s;t,x,v), \\ \dfrac{d}{ds} V(s;t,x,v) = E(s, X(s;t,x,v)) + V(s;t,x,v) \wedge B(s, X(s;t,x,v)). \end{cases}$$

Consider two solutions f_1, f_2 with $(X_1, V_1), (X_2, V_2)$ the corresponding characteristics. We study the distance

$$D(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |X_1(t,x,v) - X_2(t,x,v)| f^{in}(x,v)\, dx dv. \tag{16}$$

Additional terms in the magnetized case

$$
\begin{aligned}
D(t) &\leq \int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| \\
&\quad + |V_1(\tau) \wedge B(\tau, X_1(\tau)) - V_2(\tau) \wedge B(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds, \\
&\leq \int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds \\
&\quad + \|B\|_\infty \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_1(\tau) - V_2(\tau)| f^{in}(x,v) dx dv d\tau ds \\
&\quad + \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_2(\tau)| |B(\tau, X_1(\tau)) - B(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds, \\
&= I(t) + J(t) + K(t).
\end{aligned}
$$

The additional terms $J(t), K(t)$ are controlled with $\|\rho_1\|_p, \|\rho_2\|_p$ and velocity moments of f^{in}.

Uniqueness criterion with a general magnetic field

Theorem (R.)

Let $T > 0$ and $B := B(t, x)$ such that $B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3)$. If f^{in} satisfies

$$\forall k \geq 1, \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x, v) dxdv \leq (C_0 k)^{\frac{k}{3}}, \tag{17}$$ [16]

with C_0 a constant independent of k, then there exists at most one solution $f \in L^\infty([0\,, T], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ to the Cauchy problem for the magnetized Vlasov–Poisson system. If such a solution exists then it will verify

$$\sup_{[0\,,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{18}$$

[16] E. Miot, A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system, Comm. Math. Phys., 2016.

Propagation of moments with a general magnetic field

$$Q(t) := \sup \left\{ \int_0^t |E(s, X(s; 0, x, v))| ds, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\}, \tag{19}$$

$$N_T := \sup_{0 \le t \le T} Q(t) \le C, \tag{20}$$

$$\begin{aligned} M_k(t) &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |V(t; 0, x, v)|^k f^{in}(x, v) dv dx, \\ &\le \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v| + N_T)^k \exp(kt \|B\|_\infty) f^{in}(x, v) dv dx. \end{aligned}$$

In the unmagnetized case we have:

Theorem (Pallard [17])

Let $T > 0$, then for all $0 \le t \le T$,

$$Q(t) \le C(T^{\frac{1}{2}} + T^{\frac{7}{5}}). \tag{21}$$

[17] C. Pallard, Moment propagation for weak solutions to the Vlasov-Poisson system, Comm. Partial Differential Equations, 2012.

Estimates on the characteristics

Conjecture

For all time t such that $0 \leq t \leq T_B$,

$$Q(t) \leq C \exp(T_B \|B\|_\infty)^{\frac{2}{5}} (T_B^{\frac{1}{2}} + T_B^{\frac{7}{5}}). \tag{22}$$

- The conjecture is true if we assume $B := B(t)$ independent of x.
- Difficulty when $B := B(t,x)$: control of the difference between velocity characteristics $|V(s) - V_*(s)|$ in terms of $Q(t)$ and $|V(s) - V_*(s)|$:

$$\begin{aligned} |V(s) - V_*(s)| &\leq |v - v_*| + 2Q(t) \\ &\quad + \int_s^t |V(s) \wedge B(s, X(s)) - V_*(s) \wedge B(s, X_*(s))| ds. \end{aligned}$$

Thank you for your attention!